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Detailed wordlength pattern of regular fractional factorial split-plot designs in terms of complementary sets

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Abstract

With reference to regular fractional factorial split-plot designs, we consider a detailed wordlength pattern taking due cognizance of the distinction between the whole-plot and sub-plot factors. A generalized version of the MacWilliams' identity is employed to express the detailed wordlength pattern in terms of complementary sets. Several special features make this result intrinsically different from the corresponding one in classical fractional factorial designs where all factors have the same status. An application to robust parameter designs is indicated and examples given.

Keywords: Finite projective geometry; MacWilliams' identity; Robust parameter design; Sub-plot; Two-phase randomization; Whole-plot

1. Introduction

Fractional factorial (FF) designs are widely used to investigate the impact of factors on processes. The minimum aberration (MA) criterion ([10]) is often used to rank FF designs, and it provides a good general rule for comparing designs when all factors are of equal interest. We refer to [6] for a review, and to [5,8,16], among others, for more recent results.

The runs of a classical FF design are conducted in a completely random order. This can be impractical when it is expensive or difficult to change the levels of some of the factors because of actual physical restrictions on the process. In situations of this kind, a fractional factorial split-plot (FFSP) design, which involves a two-phase randomization, can instead be used to conveniently reduce costs and hence represents a practical design option; see [1,2,12] for details and examples.

It is, however, well known (e.g., see [2] or [13]) that the two-phase randomization in FFSP designs creates a factor asymmetry in both status and precision. Therefore, the MA criterion may be inappropriate for FFSP designs and one may be forced to consider alternate criteria to reflect an experimenter's priorities taking due notice of factor asymmetry. To that effect, in this paper, a new *detailed wordlength pattern* is proposed. Results are developed that can facilitate the application of meaningful criteria based on it.

Recent work in the classical FF setting, via consideration of complementary sets, presents a new approach for searching for optimal designs under the MA criterion ([4,5,16,18]). The use of smaller complementary sets can greatly

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ease the computational burden of the search. It is reasonable to anticipate that a similar approach can substantially simplify the identification of optimal FFSP designs as well under the appropriate criteria. The present article aims at addressing this issue. Two special features arising from the split-plot structure make the problem nontrivial. Specifically, unlike what happens in the classical FF setting, (a) two complementary sets must be accounted for; and (b) the detailed wordlength pattern of the complement alone does not uniquely determine that of the original. In particular, the handling of (b) entails much additional intricacies.

The next section presents the preliminaries including a finite geometric formulation which provides a powerful tool for the study of FFSP designs. The identities relating FFSP designs to complementary sets are developed in Section 3. Finally, in Section 4, the use of these results is demonstrated through some examples in a practical application.

2. Preliminaries

2.1. Description

We begin with a description of regular FFSP designs. This description is kept brief since an equivalent version, via a projective geometric formulation, follows in the next section. More details are available, for example, in [15].

A regular $s^{(n_1+n_2)-(p_1+p_2)}$ FFSP design involves $n = n_1 + n_2$ factors $Z_1, \ldots, Z_n$, each at s levels, where s ($\geqslant 2$) is a prime or power of prime. The levels of $Z_1, \ldots, Z_{n_1}$ ($1 \leqslant n_1 < n$) are hard to change and the remaining n_2 factors are easy to change in the course of experimentation. The design, specified via a suitable system of $(p_1 + p_2)$ independent linear equations over GF(s), involves $s^{n_1-p_1}$ distinct factor level settings of the hard to change factors $Z_1, \ldots, Z_{n_1}$. Each of these settings appears in conjunction with $s^{n_2-p_2}$ distinct settings of $Z_{n_1+1}, \ldots, Z_n$; the sets of these $s^{n_2-p_2}$ settings to be combined with different settings of $Z_1, \ldots, Z_{n_1}$ can be different. Thus, altogether the design involves $s^{(n_1+n_2)-(p_1+p_2)}$ distinct treatment combinations. Here, $0 \leqslant p_1 < n_1$, $0 \leqslant p_2 < n_2$, and $p_1 + p_2 \geqslant 1$.

The two-phase randomization of the design goes as follows. Randomly choose one of the $s^{n_1-p_1}$ distinct factor level settings of $Z_1, \ldots, Z_{n_1}$ and run the associated $s^{n_2-p_2}$ distinct combinations of $Z_{n_1+1}, \ldots, Z_n$ in a random order, keeping $Z_1, \ldots, Z_{n_1}$ fixed. Repeat this procedure for all $s^{n_1-p_1}$ selected distinct settings of $Z_1, \ldots, Z_{n_1}$. Under this kind of randomization, each of the selected $s^{n_1-p_1}$ level settings of $Z_1, \ldots, Z_{n_1}$ defines a whole-plot (WP) consisting of $s^{n_2-p_2}$ individual runs, called sub-plots (SP), obtained through variation of $Z_{n_1+1}, \ldots, Z_n$. As such, $Z_1, \ldots, Z_{n_1}$ are called WP factors and $Z_{n_1+1}, \ldots, Z_n$ are called SP factors.

As usual, a typical treatment combination, x, is denoted by an n-vector over GF(s), and a typical pencil, b, belonging to a factorial effect, is represented by a nonnull n-vector over GF(s). The first n_1 and last n_2 elements of x or b refer to WP and SP factors, respectively. Pencils with proportional elements are considered identical. A pencil with i nonzero elements represents a main effect if $i = 1$, or an i-factor interaction if $i > 1$. See [3] for details.

The defining pencils and alias sets of a regular FFSP design can be defined in the same way as classical FF designs (e.g., see [9, Chapter 8]). The minimum number of nonzero elements in a defining pencil is called the *resolution* of the design. As in the existing literature, even without explicit mentioning, we always consider only those regular FFSP designs which

(i) have resolution of at least three, and
(ii) keep every pencil representing a SP factor main effect unaliased with pencils involving only WP factors.

The requirement (i), common also for classical FF designs, ensures that no main effect pencil is a defining one and that no two distinct main effect pencils are aliased with each other. The requirement (ii) ensures that no SP factor main effect pencil is estimated at the WP level. This is important since, because of the two-phase randomization, FFSP designs have two sources of error, one at the WP level and the other at the SP level. It is not hard to show that any pencil aliased with another that involves only WP factors will be estimated with less precision than those that are not.

2.2. A projective geometric formulation

Let $t_1 = n_1 - p_1$, $t_2 = n_2 - p_2$, $t = t_1 + t_2$, and P denote the set of distinct points of the finite projective geometry PG$(t - 1, s)$. Clearly $\#P = L_t$, where # denotes cardinality and $L_u = (s^u - 1)/(s - 1)$ ($u = 0, 1, \ldots$). For $u = 1, 2, \ldots$, a $(u - 1)$-flat of P is a subset of P, with cardinality L_u, which is closed, up to proportionality, under the formation of nonnull linear combinations. Such a flat is generated by u linearly independent points of P. Let $e_1, e_2, \ldots, e_t$ be the $t \times 1$ unit vectors over GF(s), and P_1 be a $(t_1 - 1)$-flat of P generated by $e_1, e_2, \ldots, e_{t_1}$. Define P_2 as the complement of P_1 in P. For any subset C of P, let $V(C)$ be a $t \times c$ matrix with columns given by the points of C, where $c = \#C$. Also, let $\mathscr{M}(.)$ denote the column space of a matrix.

Definition 1. An ordered pair of subsets (C_1, C_2) of P is called an eligible (n_1, n_2)—pair if: (a) $\#C_i = n_i$, $i = 1, 2$, (b) $C_i \subset P_i$, $i = 1, 2$, (c) $\operatorname{rank}\{V(C_1)\} = t_1$, and (d) $\operatorname{rank}\{V(C)\} = t$, where $C = C_1 \cup C_2$.

Following [3], [15] gave a finite projective geometric formulation of regular FFSP designs. This formulation, summarized in Theorem 1 below, implies that to study such designs, it is enough to study eligible (n_1, n_2)-pairs of subsets of P.

Theorem 1. *The existence of a regular $s^{(n_1+n_2)-(p_1+p_2)}$ FFSP design is equivalent to the existence of an eligible (n_1, n_2)-pair of subsets (C_1, C_2) of P such that with $C = C_1 \cup C_2$ and*

$$V(C) = [V(C_1)|V(C_2)], \tag{2.1}$$

(i) *the treatment combinations included in the design are given by the vectors in $\mathscr{M}[V(C)^{\mathrm{T}}]$,*
(ii) *a pencil b is a defining pencil of the design if and only if $V(C)b = 0$, and*
(iii) *two distinct pencils, $b^{(1)}$ and $b^{(2)}$, neither of which is a defining pencil of the design, are aliased with each other if and only if $V(C)b^{(1)}$ and $V(C)b^{(2)}$ are proportional to the same point of P.*

Considering the cardinalities of C_1, C_2, P_1 and P_2, it follows from Theorem 1 that a regular $s^{(n_1+n_2)-(p_1+p_2)}$ FFSP design exists if and only if $n_1 \leqslant L_{t_1}$ and $n_2 \leqslant L_t - L_{t_1}$. If the equality holds in both of these places, the design is saturated and all such designs are isomorphic. Therefore, we hereafter assume that at least one of the inequalities is strict. That is, $f > 0$ where,

$$f = f_1 + f_2, \quad f_1 = L_{t_1} - n_1, \quad f_2 = L_t - L_{t_1} - n_2. \tag{2.2}$$

Consider a regular FFSP design as specified by Theorem 1 via an eligible (n_1, n_2)-pair of subsets (C_1, C_2) of P. Denote this design as $d(C_1, C_2)$. By Theorem 1(i) and (2.1), the WP and SP factors correspond to points of C_1 and C_2, respectively. For $0 \leqslant i \leqslant n_1$, $0 \leqslant j \leqslant n_2$ and $(i, j) \neq (0, 0)$, let $A_{i,j}$ be the number of distinct defining pencils of $d(C_1, C_2)$ that involve i WP and j SP factors. The quantities $\{A_{i,j}\}$, taking cognizance of the distinction between the WP and SP factors, represent the *detailed wordlength pattern* of the design $d(C_1, C_2)$. By (i) and (ii) of Section 2.1,

$$A_{i,j} = 0 \quad \text{whenever } i + j = 1 \text{ or } 2, \tag{2.3}$$

$$A_{i,1} = 0 \quad \text{for every } i \geqslant 0. \tag{2.4}$$

These are also evident from Theorem 1(ii) and Definition 1, recalling elementary properties of points and flats of a finite projective geometry. Also write

$$A_{0,0} = (s - 1)^{-1}. \tag{2.5}$$

2.3. Complementary sets: preliminaries

In order to develop a theory for the detailed wordlength pattern in terms of complementary sets, define $F_1 = P_1 - C_1$, $F_2 = P_2 - C_2$. By (2.2), F_1 and F_2 have cardinalities f_1 and f_2, respectively. Let $F = F_1 \cup F_2$, $V(F) = [V(F_1)|V(F_2)]$, and for $0 \leqslant i \leqslant f_1$, $0 \leqslant j \leqslant f_2$, define

$$\begin{aligned}\overline{A}_{i,j} = \frac{1}{s-1}\#\{b : V(F)b = 0,\ & b \text{ is } f \times 1 \text{ over GF}(s) \text{ such that among its first } f_1 \\ & \text{elements } i \text{ are nonzero and among its last } f_2 \text{ elements } j \text{ are nonzero}\},\end{aligned} \tag{2.6}$$

$$m_j = \frac{1}{s-1}\#\{b_2 : b_2 \text{ is } f_2 \times 1 \text{ over GF}(s) \text{ and has } j \text{ nonzero elements such that } V(F_2)b_2 \text{ is nonnull but proportional to some point of } P_1\}. \tag{2.7}$$

If the (F_1, F_2)-pair satisfies the rank conditions of Definition 1, then it represents an eligible (f_1, f_2)-pair and hence, by Theorem 1, a regular FFSP design. In that case, by Theorem 1(ii) and (2.6), for $(i, j) \neq (0, 0)$, $\overline{A}_{i,j}$ can be interpreted as the number of distinct defining pencils, involving i WP factors and j SP factors, of such a complementary FFSP design. Similarly, then by Theorem 1(iii) and (2.7) for $j \geqslant 2$, m_j can be interpreted as the number of distinct pencils involving j SP factors alone that are aliased with pencils involving WP factors alone in the complementary design. In general, however, (F_1, F_2) may not satisfy the rank conditions of Definition 1. Still, even without an interpretation just discussed, the quantities $\overline{A}_{i,j}$ and m_j are well defined via (2.6) and (2.7).

Analogous to (2.3)–(2.5), from (2.6) and (2.7), one gets

$$\overline{A}_{i,j} = 0 \quad \text{whenever } i + j = 1 \text{ or } 2, \tag{2.8}$$

$$\overline{A}_{i,1} = 0 \quad \text{for every } i \geqslant 0, \tag{2.9}$$

$$\overline{A}_{0,0} = (s-1)^{-1} \text{ and} \tag{2.10}$$

$$m_0 = m_1 = 0. \tag{2.11}$$

In Section 3, the detailed wordlength pattern of $d(C_1, C_2)$ will be expressed in terms of the characteristics $\overline{A}_{i,j}$ and m_j of the complementary sets. This will substantially simplify the task of identifying optimal designs in the practically important nearly saturated cases when f_1 and f_2 are small and hence F_1 and F_2 are easy to handle.

2.4. A useful result

Before concluding this section, we present a generalized version of the MacWilliams' identity that plays an important role in the derivation of our main result. Recall that the finite Euclidean geometry $\mathrm{EG}(n, s)$ consists of all possible n-vectors over $\mathrm{GF}(s)$. For any $u = (u_1, \ldots, u_n)^{\mathrm{T}} \in \mathrm{EG}(n, s)$, where $n = n_1 + n_2$, define the "left" and "right" weights

$$\mathrm{wt_L}(u) = \#\{i : u_i \neq 0, 1 \leqslant i \leqslant n_1\} \quad \mathrm{wt_R}(u) = \#\{i : u_i \neq 0, n_1 + 1 \leqslant i \leqslant n\}.$$

For any linear subspace H of $\mathrm{EG}(n, s)$, define the weight enumerator

$$W_H(y_1, y_2) = \sum_{u \in H} y_1^{\mathrm{wt_L}(u)} y_2^{\mathrm{wt_R}(u)} = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} G_{i,j} y_1^i y_2^j, \tag{2.12}$$

where

$$G_{i,j} = \#\{u : u \in H, \mathrm{wt_L}(u) = i, \mathrm{wt_R}(u) = j\} \quad 0 \leqslant i \leqslant n_1, 0 \leqslant j \leqslant n_2. \tag{2.13}$$

Similarly, let

$$W_{H^\perp}(y_1, y_2) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} G_{i,j}^\perp y_1^i y_2^j \tag{2.14}$$

be defined with respect to the orthogonal complement $H^\perp$ of H in $\mathrm{EG}(n, s)$, where $G_{i,j}^\perp$ are defined as in (2.13), with H being replaced by $H^\perp$. Then the following result holds (see [14, Chapter 5]).

Theorem 2.

(a) $W_{H^\perp}(y_1, y_2) = (1/\#H)\{1 + (s-1)y_1\}^{n_1}\{1 + (s-1)y_2\}^{n_2} W_H((1-y_1)/(1+(s-1)y_1), (1-y_2)/(1+(s-1)y_2))$,
(b) $(1/\#H) W_H(y_1, y_2) = s^{-n}\{1 + (s-1)y_1\}^{n_1}\{1 + (s-1)y_2\}^{n_2} W_{H^\perp}((1-y_1)/(1+(s-1)y_1), (1-y_2)/(1+(s-1)y_2))$.

Remark 1. It can be verified that the conclusion of Theorem 2 remains valid if, instead of being a linear subspace of $\mathrm{EG}(n, s)$, H is an l-fold repetition of a linear subspace for any positive integer l. This interpretation is useful later. Note, however, that each point of $\mathrm{EG}(n, s)$, which is orthogonal to every member of H, is counted exactly once in $H^{\perp}$ as used in Theorem 2.

3. Main result

3.1. Statement and implications

We begin by introducing some notation. Let

$$\theta = s^{t-1}, \quad \theta_1 = s^{t_1-1}, \quad \theta_2 = \theta - \theta_1. \tag{3.1}$$

For $0 \leqslant \mu \leqslant n_1$, $0 \leqslant v \leqslant n_2$, $0 \leqslant i \leqslant f_1$, and $0 \leqslant j \leqslant f_2$, define

$$\delta_1(\mu) = \binom{n_1}{\mu}(s-1)^{\mu}, \quad \delta_2(\mu) = \sum_{r=0}^{\mu}(-1)^r\binom{\theta_1}{r}\binom{n_1-\theta_1}{\mu-r}(s-1)^{\mu-r}, \tag{3.2}$$

$$\rho_1(\mu, i) = \sum_1 \binom{\theta_1 - f_1}{r_1}\binom{n_1-\theta_1}{r_2}\binom{f_1-i}{r_3}(-1)^{r_1}(s-1)^{r_2}(s-2)^{r_3}, \tag{3.3}$$

$$\rho_2(v, j) = \sum_2 \binom{\theta_2 - f_2}{r_1}\binom{n_2-\theta_2}{r_2}\binom{f_2-j}{r_3}(-1)^{r_1}(s-1)^{r_2}(s-2)^{r_3}, \tag{3.4}$$

$$\rho_3(v, j) = \sum_2 \binom{\theta - f_2}{r_1}\binom{n_2-\theta}{r_2}\binom{f_2-j}{r_3}(-1)^{r_1}(s-1)^{r_2}(s-2)^{r_3} \tag{3.5}$$

with $\sum_1$ and $\sum_2$ denoting sums over nonnegative integers r_1–r_3 satisfying

$$r_1 + r_2 + r_3 = \mu - i \quad \text{and} \quad r_1 + r_2 + r_3 = v - j, \tag{3.6}$$

respectively. In the above, for any integer, a, and any nonnegative integer, r, we define $\binom{a}{r}$ as $a(a-1)\cdots(a-r+1)/r!$ when $r \geqslant 1$ or 1 if $r = 0$. Our main result is the following theorem that expresses the $A_{\mu,v}$ involved in the detailed wordlength pattern in terms of complementary sets.

Theorem 3. *For $0 \leqslant \mu \leqslant n_1$, $0 \leqslant v \leqslant n_2$, with reference to the regular FFSP design $d(C_1, C_2)$,*

$$A_{\mu,v} = \text{constant} + \sum_{i=0}^{f_1}\sum_{j=0}^{f_2}(-1)^{(i+j)}\rho_1(\mu, i)\rho_2(v, j)\overline{A}_{i,j}$$
$$+ \frac{1}{s^{t_1}}\sum_{j=0}^{f_2}(-1)^j\{\delta_1(\mu)\rho_3(v, j) - \delta_2(\mu)\rho_2(v, j)\}(\overline{A}_{0,j} + m_j),$$

where the constant may depend on μ, v, s, n_1, n_2, p_1 or p_2, but not on the choice of C_1 or C_2.

We first discuss the implications of Theorem 3 and delay the proof until the next subsection. It is important to note that in terms of applications, this theorem is much simpler than it seems. This happens because $\rho_1(\mu, i) = 0$ for $i > \mu$ and $\rho_2(v, j) = \rho_3(v, j) = 0$ for $j > v$, in light of (3.3)–(3.6). Further simplification arises because of (2.8)–(2.11). Moreover, (3.3)–(3.5) become considerably simpler for $s = 2$. On the basis of these considerations, the following identities emerge from Theorem 3 after some algebra. These are very helpful in applications, and several examples are given in Section 4.

$$A_{0,3} = \text{constant} - \overline{A}_{0,3} - m_2, \tag{3.7}$$

$$A_{1,2} = \text{constant} - \overline{A}_{1,2} + m_2, \tag{3.8}$$

$$A_{3,0} = \text{constant} - \overline{A}_{3,0}, \tag{3.9}$$

$$A_{0,4} = \text{constant} + (3s-5)\overline{A}_{0,3} + \overline{A}_{0,4} + \tfrac{1}{2}\{s^{t_1} + 5(s-2)\}m_2 + m_3, \tag{3.10}$$

$$A_{1,3} = \text{constant} + 2(s-2)\overline{A}_{1,2} + \overline{A}_{1,3} - \{n_1(s-1) + 2(s-2)\}m_2 - m_3, \tag{3.11}$$

$$A_{2,2} = \text{constant} + (s-1)\overline{A}_{1,2} + \overline{A}_{2,2} - \tfrac{1}{2}\{s^{t_1} - 2n_1(s-1) + s - 2\}m_2, \tag{3.12}$$

$$A_{4,0} = \text{constant} + (3s-5)\overline{A}_{3,0} + \overline{A}_{4,0}. \tag{3.13}$$

Remark 2. In (3.7)–(3.13), any $\overline{A}_{i,j}$ is interpreted as zero if $i > f_1$ or $j > f_2$. Similarly, any m_j is interpreted as zero if $j > f_2$.

As foreshadowed in the introduction, Theorem 3 reveals that the $\overline{A}_{i,j}$ alone do not uniquely determine the $A_{\mu,\nu}$; one needs to consider the m_j's as well. The following example showing two regular FFSP designs with the same $\overline{A}_{i,j}$ but different $A_{\mu,\nu}$ re-emphasizes this point.

Example 1. Consider two regular $2^{(2+9)-(0+7)}$ FFSP designs d_1 and d_2 such that for both of them $C_1 = \{2, 12\}$, while C_2 equals $\{13, 4, 14, 24, 124, 34, 134, 234, 1234\}$ for d_1, and $\{3, 13, 14, 24, 124, 34, 134, 234, 1234\}$ for d_2. For notational simplicity, a typical point $(j_1, \ldots, j_4)'$ of PG(3, 2) is denoted here by $1^{j_1} \cdots 4^{j_4}$ with the convention that i^{j_i} is dropped when $j_i = 0$. It can be checked that d_1 has the same set of $\overline{A}_{i,j}$'s as d_2. However, d_1 and d_2 have different $A_{\mu,\nu}$'s. For example, $A_{1,2}$ equals 8 for d_1 and 6 for d_2. On the other hand, $A_{0,3}$ equals 4 for d_1 and 6 for d_2.

3.2. *Proof of Theorem 3*

For ease of presentation, Theorem 3 is proved in several steps. In particular, Steps IV and V reveal the speciality of the regular FFSP setting.

Step I: With reference to a regular FFSP design, $d(C_1, C_2)$, let $H = \mathscr{M}[V(C)^{\mathrm{T}}]$, where $C = C_1 \cup C_2$. By condition (d) of Definition 1, $V(C)$ has full row rank. Hence H is a linear subspace of EG(n, s) and $\#H = s^t$. Let $H^\perp$ be the orthogonal complement of H in EG(n, s). Define the quantities $G_{i,j}$ and $G^\perp_{i,j}$, with respect to H and $H^\perp$, respectively, as in Section 2.4. By Theorem 1(ii) and the definition of $A_{i,j}$, clearly $A_{i,j} = (s-1)^{-1}G^\perp_{i,j}$ for every i, j. Hence, by Theorem 2(a)

$$\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} A_{i,j}y_1^i y_2^j = \frac{1}{(s-1)s^t}\{1+(s-1)y_1\}^{n_1}\{1+(s-1)y_2\}^{n_2} W_H\left(\frac{1-y_1}{1+(s-1)y_1}, \frac{1-y_2}{1+(s-1)y_2}\right). \tag{3.14}$$

Step II: By the definition of F_1, F_2 and F, $V(P) = [V(C)|V(F)]$. Let Q be a matrix of order $s^t \times L_t$ with rows given by the s^t vectors in the row space of $V(P)$. Partition Q as

$$Q = [Q_1|Q_2], \tag{3.15}$$

where Q_1 is $s^t \times n$ and Q_2 is $s^t \times f$; recall that $n + f = L_t$, by (2.2). Evidently, the rows of Q_1 and Q_2 are generated by the rows of $V(C)$ and $V(F)$, respectively. Thus the columns in Q_1^{T} constitute the linear subspace H. Let $\overline{H}$ be the set of columns in Q_2^{T}. If $V(F)$ has full row rank, then $\overline{H}$ is also a linear subspace of EG(n, s); otherwise it is an s^{t-r}−fold repetition of a linear subspace where $r = \text{rank}\{V(F)\}$. Let $\overline{H}^\perp$ be the orthogonal complement of $\overline{H}$ in EG(n, s). With reference to $\overline{H}$ and $\overline{H}^\perp$, define the quantities $\overline{G}_{i,j}$ and $\overline{G}^\perp_{i,j}$ in the same way as $G_{i,j}$ and $G^\perp_{i,j}$ were defined with respect to H and $H^{\perp}$, the only difference being that f_1 and f_2 now play the roles of n_1 and n_2. Then by (2.6), $\overline{A}_{i,j} = (s-1)^{-1}\overline{G}^{\perp}_{i,j}$. Recalling Remark 1 and using Theorem 2(b), with H there replaced by $\overline{H}$, one now gets

$$\frac{1}{s^t} W_{\overline{H}}(v_1, v_2) = s^{-f}(s-1) \sum_{i=0}^{f_1} \sum_{j=0}^{f_2} \overline{A}_{i,j}(1-v_1)^i\{1+(s-1)v_1\}^{f_1-i}(1-v_2)^j\{1+(s-1)v_2\}^{f_2-j}. \quad (3.16)$$

Step III: By (3.15) and the definitions of H and $\overline{H}$, any row of Q is of the form $(u^{\mathrm{T}}, \overline{u}^{\mathrm{T}})$, where $u \in H, \overline{u} \in \overline{H}$. Since $V(P)$ has full row rank, only one row of Q equals the null vector. Call this row $(u_0^{\mathrm{T}}, \overline{u}_0^{\mathrm{T}})$. Furthermore, by well-known results on orthogonal arrays, the rows of Q form a *saturated* orthogonal array $\mathrm{OA}(s^t, L_t, s, 2)$ (e.g., see [11]), and hence every nonnull row of Q has $\theta = s^{t-1}$ nonzero elements (cf. Lemma 3.1 of [7]). Therefore,

$$\mathrm{wt}(u_0) = \mathrm{wt}(\overline{u}_0) = 0, \quad (3.17)$$

and for every other row $(u^{\mathrm{T}}, \overline{u}^{\mathrm{T}})$ of Q

$$\mathrm{wt}(u) + \mathrm{wt}(\overline{u}) = \theta, \quad (3.18)$$

where, as usual, $\mathrm{wt}(u)$ and $\mathrm{wt}(\overline{u})$ denote the numbers of nonzero elements of u and $\overline{u}$, respectively.

Step IV: For each $u \in H$, define $\mathrm{wt_L}(u)$ and $\mathrm{wt_R}(u)$ as in Section 2.4. Similarly, for $\overline{u} \in \overline{H}$, define $\mathrm{wt_L}(\overline{u})$ and $\mathrm{wt_R}(\overline{u})$, replacing n_1, n_2 with f_1, f_2. Now, recall that $C = C_1 \cup C_2$ and $F = F_1 \cup F_2$. Hence Q_1 and Q_2 of (3.15) can be partitioned as

$$Q_1 = [Q_{11}|Q_{12}], \quad Q_2 = [Q_{21}|Q_{22}],$$

where Q_{11} ,Q_{12}, Q_{21} ,Q_{22} correspond to $V(C_1)$, $V(C_2)$, $V(F_1)$, $V(F_2)$, respectively. Since $C_1 \cup F_1 = P_1$, which is a (t_1-1)-flat of P, it is clear that the rows of $Q^{(1)} = [Q_{11}|Q_{21}]$ constitute an s^{t-t_1} $(=s^{t_2})$-fold repetition of a *saturated* orthogonal array $\mathrm{OA}(s^{t_1}, L_{t_1}, s, 2)$. Thus as before $Q^{(1)}$ has s^{t_2} null rows and every nonnull row of $Q^{(1)}$ has exactly $\theta_1 = s^{t_1-1}$ nonzero elements. Let N denote the set of points of H associated with the s^{t_2} null rows of $Q^{(1)}$. Since one of the null rows of $Q^{(1)}$ corresponds to the null row of $(u_0^{\mathrm{T}}, \overline{u}_0^{\mathrm{T}})$ of Q, clearly $u_0 \in N$. As before , for any $u \in H$, let $\overline{u} \in \overline{H}$ be such that $(u^{\mathrm{T}}, \overline{u}^{\mathrm{T}})$ constitutes a row of Q. Then from the above discussion,

$$\mathrm{wt_L}(u) = \mathrm{wt_L}(\overline{u}) = 0 \quad \text{for all } u \in N,$$

$$\mathrm{wt_L}(u) + \mathrm{wt_L}(\overline{u}) = \theta_1 \quad \text{for all } u \in H - N.$$

Let $\theta_2 = \theta - \theta_1 = s^{t-1} - s^{t_1-1}$. Then combining the above with (3.17) and (3.18),

$$\mathrm{wt_L}(u_0) = \mathrm{wt_R}(u_0) = \mathrm{wt_L}(\overline{u}_0) = \mathrm{wt_R}(\overline{u}_0) = 0, \quad (3.19)$$

$$\mathrm{wt_L}(u) = \mathrm{wt_L}(\overline{u}) = 0, \quad \mathrm{wt_R}(u) + \mathrm{wt_R}(\overline{u}) = \theta \text{ for all } u (\neq u_0) \in N, \quad (3.20)$$

$$\mathrm{wt_L}(u) + \mathrm{wt_L}(\overline{u}) = \theta_1, \quad \mathrm{wt_R}(u) + \mathrm{wt_R}(\overline{u}) = \theta_2 \text{ for all } u \in H - N. \quad (3.21)$$

By (2.12) and (3.19)–(3.21), after some algebra,

$$W_H(z_1, z_2) = z_1^{\theta_1} z_2^{\theta_2} W_{\overline{H}}\left(\frac{1}{z_1}, \frac{1}{z_2}\right) + z_2^{\theta_2}(z_2^{\theta_1} - z_1^{\theta_1}) \sum_{u \in N} \left(\frac{1}{z_2}\right)^{\mathrm{wt_R}(\overline{u})} + 1 - z_2^{\theta}. \quad (3.22)$$

Step V: We now consider the second term on the right-hand side of (3.22). The following considerations help. For any $u \in N$, let $h = h(\overline{u})$ be a subvector of $\overline{u}$ given by the last f_2 elements of $\overline{u}$. Define $\overline{N} = \{h : h = h(\overline{u}), u \in N\}$. Observe that

(i) $\overline{N}$ consists of the transposes of those rows of Q_{22} which correspond to the null rows of $Q^{(1)} = [Q_{11}|Q_{21}]$,
(ii) the rows of Q_{22} are given by all possible linear combinations of the rows of $V(F_2)$, and
(iii) the rows of $Q^{(1)}$ are given by all possible linear combinations of the rows of $V(P_1) = [V(C_1)|V(F_1)]$.

Since P_1 is spanned by the unit vectors $e_1, \ldots, e_{t_1}$ over GF(s), the first t_1 rows of $V(P_1)$ are linearly independent and the remaining $t - t_1$ $(=t_2)$ rows of $V(P_1)$ are null vectors. Therefore by (iii) above, the s^{t_2} null rows of $Q^{(1)}$ are given by all possible linear combinations of the last t_2 rows of $V(P_1)$. Consequently, partitioning $V(F_2)$ as

$$V(F_2) = \begin{bmatrix} V_1(F_2) \\ V_2(F_2) \end{bmatrix},$$

where $V_i(F_2)$ has t_i rows $(i = 1, 2)$, it follows from (i) and (ii) above that $\overline{N}$ consists of the transposes of the s^{t_2} possible linear combinations of the rows of $V_2(F_2)$. Thus $\overline{N}$ is a linear subspace of EG(f_2, s) if $V_2(F_2)$ has full row rank; otherwise $\overline{N}$ is an s^{t_2-r}-fold repetition of a linear subspace, where $r = \text{rank}\{V_2(F_2)\}$. Let $\overline{N}^{\perp}$ be the orthogonal complement of $\overline{N}$ in EG(f_2, s), and for $0 \leqslant j \leqslant f_2$, let ψ_j and $\psi_j^{\perp}$ denote, respectively, the numbers of members of $\overline{N}$ and $\overline{N}^{\perp}$ having weight j. Then by the usual form of the MacWilliams' identity, noting that $\#\overline{N} = s^{t_2}$,

$$\sum_{j=0}^{f_2} \psi_j y^j = \frac{1}{s^{f_2-t_2}} \sum_{j=0}^{f_2} \psi_j^{\perp} (1-y)^j \{1 + (s-1)y\}^{f_2-j}. \tag{3.23}$$

Observe also that for any $f_2 \times 1$ vector b_2 over GF(s), $V_2(F_2)b_2 = 0$ if and only if $V(F_2)b_2$ is either null or nonnull but proportional to some point of P_1. Consequently by (2.6), (2.7) and the definition of $\psi_j^{\perp}$,

$$\psi_j^{\perp} = (s-1)(\overline{A}_{0,j} + m_j), \quad 0 \leqslant j \leqslant f_2. \tag{3.24}$$

We now return to the second term on the right-hand side of (3.22). For $u \in N$ note that $\text{wt}_{\text{R}}(\overline{u}) = \text{wt}(h)$, where $h = h(\overline{u})$ and $\overline{u}$ corresponds to u. Hence by (3.23) and (3.24),

$$\begin{aligned} \sum_{u \in N} \left(\frac{1}{z_2}\right)^{\text{wt}_{\text{R}}(\overline{u})} &= \sum_{h \in \overline{N}} \left(\frac{1}{z_2}\right)^{\text{wt}(h)} = \sum_{j=0}^{f_2} \psi_j \left(\frac{1}{z_2}\right)^j \\ &= \frac{(s-1)}{s^{f_2-t_2}} \sum_{j=0}^{f_2} (\overline{A}_{0,j} + m_j) \left(1 - \frac{1}{z_2}\right)^j \left(1 + \frac{s-1}{z_2}\right)^{f_2-j}. \end{aligned} \tag{3.25}$$

Step VI: Let

$$B(y_1, y_2) = \frac{\{1 + (s-1)y_1\}^{n_1} \{1 + (s-1)y_2\}^{n_2}}{(s-1)s^t}, \tag{3.26}$$

$$z_i = \frac{1 - y_i}{1 + (s-1)y_i}, \quad (i = 1, 2). \tag{3.27}$$

Then by (3.14), (3.16), (3.22) and (3.25)

$$\begin{aligned} &\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} A_{i,j} y_1^i y_2^j \\ &\quad = B(y_1, y_2) \Bigg[z_1^{\theta_1} z_2^{\theta_2} \left(\frac{s-1}{s^{f-t}}\right) \sum_{i=0}^{f_1} \sum_{j=0}^{f_2} \overline{A}_{i,j} \left(1 - \frac{1}{z_1}\right)^i \left(1 + \frac{s-1}{z_1}\right)^{f_1-i} \left(1 - \frac{1}{z_2}\right)^j \left(1 + \frac{s-1}{z_2}\right)^{f_2-j} \\ &\qquad + z_2^{\theta_2} (z_2^{\theta_1} - z_1^{\theta_1}) \left(\frac{s-1}{s^{f_2-t_2}}\right) \sum_{j=0}^{f_2} (\overline{A}_{0,j} + m_j) \left(1 - \frac{1}{z_2}\right)^j \left(1 + \frac{s-1}{z_2}\right)^{f_2-j} + 1 - z_2^{\theta} \Bigg]. \end{aligned}$$

Simplification using (3.26) and (3.27) yields

$$\begin{aligned}
&\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} A_{i,j} y_1^i y_2^j \\
&\quad = (1-y_1)^{\theta_1-f_1}\{1+(s-1)y_1\}^{n_1-\theta_1}(1-y_2)^{\theta_2-f_2}\{1+(s-1)y_2\}^{n_2-\theta_2} \\
&\qquad \times \sum_{i=0}^{f_1}\sum_{j=0}^{f_2}(-1)^{i+j}\overline{A}_{i,j} y_1^i\{1+(s-2)y_1\}^{f_1-i} y_2^j\{1+(s-2)y_2\}^{f_2-j} \\
&\qquad + \frac{1}{s^{t_1}}\Bigg[\Big\{(1+(s-1)y_1)^{n_1}(1-y_2)^{\theta-f_2}(1+(s-1)y_2)^{n_2-\theta} \\
&\qquad -(1-y_1)^{\theta_1}(1+(s-1)y_1)^{n_1-\theta_1}(1-y_2)^{\theta_2-f_2}(1+(s-1)y_2)^{n_2-\theta_2}\Big\} \\
&\qquad \times \sum_{j=0}^{f_2}(-1)^j(\overline{A}_{0,j}+m_j)y_2^j\{1+(s-2)y_2\}^{f_2-j}\Bigg] \\
&\qquad + \frac{1}{(s-1)s^t}\{1+(s-1)y_1\}^{n_1}[\{1+(s-1)y_2\}^{n_2}-(1-y_2)^{\theta}\{1+(s-1)y_2\}^{n_2-\theta}].
\end{aligned}$$

Equating coefficients of $y_1^{\mu} y_2^{\nu}$ from both sides, Theorem 3 follows.

4. Applications

4.1. Design criterion

The identities of Section 3.1 provide a powerful tool for studying regular FFSP designs. The attractive feature of the detailed wordlength pattern is its flexibility, and, because of this, it lends itself to easy adaptation to many important applied problems. It is the fact that one can and may so easily change the ordering of the $A_{i,j}$'s and find optimal designs for specific applications that makes the developments in Section 3 more important than simply giving a list of designs under one criterion. For the purpose of illustration, we demonstrate the suppleness of the approach on the important industrial problem of robust parameter design.

Robust parameter design (RPD) is an approach to planned experimentation for reducing the variation of a process [17]. The goal of such experiments is to adjust the levels of *control factors* so that the process variation due to changes in hard to control *noise factors* is minimized. This is a fundamentally important industrial problem which has received much attention in recent years (see [19, Chapter 10], and references therein). In RPD's, the effects of primary interest are the control factor main effects, the control-by-noise factor interactions and control-by-control factor interactions. Identification of these effects gives the experimenter the potential to improve the process by adjusting the mean with the significant control effects and dampening the effect of the noise factors. Noise-by-noise factor interactions are of less interest. Since the levels of noise factors cannot be set in everyday practice, they cannot be directly used to improve the process. As is the usual convention in fractionated experiments, interactions among three or more factors are hereafter assumed to be negligible.

Suppose an experiment is to be performed with the hard to change noise factors as WP factors and control factors as SP factors. In this situation, the detailed wordlength pattern is particularly powerful because the defining pencils with the same number of nonzero elements have different meanings. For instance, $A_{3,0}$ relates to defining pencils involving only noise factors and entails the sacrifice of only some noise-by-noise factor interactions to estimate noise main effects. However, $A_{1,2}$ aliases control-by-noise factor interactions with main effects. This is quite severe since estimation of control-by-noise interactions is a primary objective of the experiment. As a result, a smaller $A_{1,2}$ is more desirable than a smaller $A_{3,0}$ in order that less information about the effects of interest is sacrificed. Furthermore, for the same reason, we also argue that a smaller $A_{2,2}$, $A_{1,3}$ or $A_{0,4}$ is more desirable than a smaller $A_{3,0}$.

Being cognizant that the design is a FFSP, we note that a smaller $A_{2,2}$ is more desirable than a smaller $A_{0,4}$, since unlike $A_{0,4}$, $A_{2,2}$ entails the aliasing of pencils involving some control (SP) factors with pencils involving noise (WP) factors alone and thus the assessment of the former with the larger error variance. Following this line of reasoning, it makes sense to consider an optimality criterion that calls for the sequential minimization of

$$\{A_{1,2}, A_{0,3}, A_{2,2}, A_{1,3}, A_{0,4}, A_{3,0}, A_{4,0}, \ldots\}. \tag{4.1}$$

Recall that by (2.4) $A_{i,1} = 0$ for all $i \geqslant 0$.

4.2. Examples and tables

Considering the criterion given in (4.1), we now present several examples showing the application of the formulae in Section 3.1. We emphasize that this criterion is chosen for the sake of illustration, and that the identities can work equally well under other criteria, arising in other situations.

In what follows, $e_1, e_2, \ldots, e_t$ continue to denote the $t \times 1$ unit vectors over GF(s). Thus, $e_1, e_2, \ldots, e_{t_1}$ span P_1 and $e_1, e_2, \ldots, e_t$ span P. We also write $\xi_i = e_{t_1+i}$ $(i = 1, 2, \ldots)$.

Example 2. Let $f_1 = 0$, $f_2 = 3$, and $s = 3$.

(a) First suppose $t_2 = 1$. Then F_2 must be of the form $\{\xi_1, \xi_1 + \lambda_1 a_1, \xi_1 + \lambda_2 a_2\}$, where $\lambda_1, \lambda_2 \in \{1, 2\}$ and a_1, a_2 are points of P_1. Then $m_2 = 3$, and as $\overline{A}_{1,2} = 0$ (see Remark 2) by (3.8), all designs have the same $A_{1,2}$. Next consider $A_{0,3}$. Since $m_2 = 3$ for all designs, by (3.7) in order to minimize $A_{0,3}$, we need to maximize $\overline{A}_{0,3}$. Up to isomorphism, the unique choice of F_2 achieving this is $\{\xi_1, \xi_1 + e_1, \xi_1 + 2e_1\}$. This gives the optimal design under the present criterion.

(b) Next let $t_2 \geqslant 2$. By Remark 2, we still have $\overline{A}_{1,2} = 0$ for all designs. Hence by(3.8), minimization of $A_{1,2}$ calls for minimization of m_2. Up to isomorphism, the only choices of F_2 making $m_2 = 0$ are: (i) $\{\xi_1, \xi_2, \xi_1 + \xi_2\}$, (ii) $\{\xi_1, \xi_2, \xi_1 + \xi_2 + e_1\}$, (iii) $\{\xi_1, \xi_2, \xi_3\}$. Of these, (iii) can only arise when $t_2 \geqslant 3$. All of these possibilities yield the same $A_{1,2}$ by (3.8). However, $\overline{A}_{0,3} = 1$ for (i) and $\overline{A}_{0,3} = 0$ for (ii) and (iii). Hence, considering $A_{0,3}$ next, by (3.7) the choice (i) gives the optimal design.

Example 3. Let $f_1 = 2$, $f_2 = 2$. Up to isomorphism, one can take $F_1 = \{e_1, e_2\}$ and consider the following choices of F_2: (i) $\{\xi_1, \xi_1 + e_1\}$, (ii) $\{\xi_1, \xi_1 + e_1 + e_2\}$, (iii) $\{\xi_1, \xi_1 + e_3\}$, (iv) $\{\xi_1, \xi_2\}$. Of these, (iii) and (iv) can arise only when $t_1 \geqslant 3$ and $t_2 \geqslant 2$, respectively. For (i)–(iv), the pair $(\overline{A}_{1,2}, m_2)$ equals (1, 1), (0, 1), (0, 1) and (0, 0), respectively. Hence by (3.8), consideration of $A_{1,2}$ eliminates (ii) and (iii). Also, by Remark 2, here $\overline{A}_{0,3} = 0$ for all designs. Consequently by (3.7), consideration of $A_{0,3}$ eliminates (iv). Thus the optimal design is given by $F_1 = \{e_1, e_2\}$, $F_2 = \{\xi_1, \xi_1 + e_1\}$.

Example 4. Let $f_1 = 1$, $f_2 = 3$ and $s = 2$. Suppose $t_1 \geqslant 2$ and $t_2 \geqslant 2$. This covers, in particular, the case of Example 1. Up to isomorphism, there are seven possible choices of F_1 and F_2:

(i) $F_1 = \{e_1\}$, $F_2 = \{\xi_1, \xi_1 + e_1, \xi_1 + e_2\}$,
(ii) $F_1 = \{e_1\}$, $F_2 = \{\xi_1, \xi_1 + e_1, \xi_2\}$,
(iii) $F_1 = \{e_1\}$, $F_2 = \{\xi_1, \xi_2, \xi_1 + \xi_2\}$,
(iv) $F_1 = \{e_1\}$, $F_2 = \{\xi_1, \xi_2, \xi_1 + e_2\}$,
(v) $F_1 = \{e_1\}$, $F_2 = \{\xi_1, \xi_2, \xi_1 + \xi_2 + e_1\}$,
(vi) $F_1 = \{e_1\}$, $F_2 = \{\xi_1, \xi_2, \xi_1 + \xi_2 + e_2\}$,
(vii) $F_1 = \{e_1\}$, $F_2 = \{\xi_1, \xi_2, \xi_3\}$.

Of these, (vii) can arise only when $t_2 \geqslant 3$. For (i)–(vii) above, the pair $(\overline{A}_{1,2}, m_2)$ equals (1, 3), (1, 1), (0, 0), (0, 1), (0, 0), (0, 0), (0, 0), respectively. Therefore, by (3.8), consideration of $A_{1,2}$ eliminates (i) and (iv). For (ii), (iii), (v)–(vii), the pair $(\overline{A}_{0,3}, m_2)$ equals (0, 1), (1, 0), (0, 0), (0, 0) and (0, 0), respectively. Hence by (3.7), consideration of $A_{0,3}$ eliminates (v)–(vii). Continuing, we next compare (ii) and (iii) on the basis of $A_{2,2}$. By Remark 2, $\overline{A}_{2,2} = 0$ for all designs. Taking $s = 2$ in (3.12),

$$A_{2,2} = \text{constant} + \overline{A}_{1,2} + (n_1 - 2^{t_1 - 1})m_2.$$

Table 1
Optimal 16-run FFSP designs for $s = 2$ and $n_1 + n_2 \leqslant 7$

(n_1, n_2, p_1, p_2)	C_1	C_2
(1, 4, 0, 1)	{1}	{2, 3, 4, 1234}
(2, 3, 0, 1)	{1, 2}	{3, 4, 1234}
(3, 2, 1, 0)	{1, 2, 12}	{3, 4}
(3, 2, 0, 1)	{1, 2, 3}	{4, 1234}
(4, 1, 1, 0)	{1, 2, 3, 123}	{4}
(1, 5, 0, 2)	{1}	{2, 3, 4, 123, 124}
(2, 4, 0, 2)	{1, 2}	{3, 4, 123, 134}
(3, 3, 1, 1)	{1, 2, 12}	{3, 4, 134}
(3, 3, 0, 2)	{1, 2, 3}	{4, 124, 134}
(4, 2, 1, 1)	{1, 2, 3, 12}	{4, 134}
(5, 1, 2, 0)	{1, 2, 3, 12, 13}	{4}
(1, 6, 0, 3)	{1}	{2, 3, 4, 123, 124, 134}
(2, 5, 0, 3)	{1, 2}	{3, 4, 123, 124, 134}
(3, 4, 1, 2)	{1, 2, 12}	{3, 4, 13, 234}
(3, 4, 0, 3)	{1, 2, 3}	{4, 124, 134, 234}
(4, 3, 1, 2)	{1, 2, 3, 123}	{4, 124, 134}
(5, 2, 2, 1)	{1, 2, 3, 12, 13}	{4, 234}
(6, 1, 3, 0)	{1, 2, 3, 12, 13, 23}	{4}

Table 2
Optimal 27-run FFSP designs for $s = 3$ and $n_1 + n_2 \leqslant 7$

(n_1, n_2, p_1, p_2)	C_1	C_2
(1, 3, 0, 1)	{1}	{2, 3, 123}
(2, 2, 0, 1)	{1, 2}	{3, 123}
(3, 1, 1, 0)	{1, 2, 12}	{3}
(1, 4, 0, 2)	{1}	{2, 3, 23, 12^23}
(2, 3, 0, 2)	{1, 2}	{3, 12^23, 12^23^2}
(3, 2, 1, 1)	{1, 2, 12}	{3, 12^23}
(4, 1, 2, 0)	{1, 2, 12, 12^2}	{3}
(1, 5, 0, 3)	{1}	{2, 3, 12, 12^23, 12^23^2}
(2, 4, 0, 3)	{1, 2}	{3, 13, 123^2, 12^23^2}
(3, 3, 1, 2)	{1, 2, 12}	{3, 12^23, 12^23^2}
(4, 2, 2, 1)	{1, 2, 12, 12^2}	{3, 13}
(1, 6, 0, 4)	{1}	{2, 3, 12, 13^2, 23^2, 12^23^2}
(2, 5, 0, 4)	{1, 2}	{3, 13^2, 23, 123, 123^2}
(3, 4, 1, 3)	{1, 2, 12}	{3, 13, 23, 123^2}
(4, 3, 2, 2)	{1, 2, 12, 12^2}	{3, 13, 23}

But by (2.2), $n_1 = L_{t_1} - f_1 = 2^{t_1} - 2$. Therefore, $A_{2,2} = \text{constant} + \overline{A}_{1,2} + (2^{t_1-1} - 2)m_2$. Noting that $t_1 \geqslant 2$, it follows that $A_{2,2}$ for (ii) is larger than $A_{2,2}$ for (iii). Hence (iii) represents the optimal design under our criterion.

For ready reference to the reader, optimal FFSP designs, with respect to sequential minimization of (4.1), are shown in Tables 1 and 2 for $n_1 + n_2 \leqslant 7$. Both tables show sets C_1 and C_2 for the optimal designs. Table 1 contains the 16-run designs for $s = 2$ and Table 2 presents 27-run designs for $s = 3$. In both tables, a typical point $(\zeta_1, \ldots, \zeta_t)'$ of $\mathrm{PG}(t-1, s)$ is represented using the compact notation $1^{\zeta_1} \ldots t^{\zeta_t}$, where i^{ζ_i} is dropped if $\zeta_i = 0$ (e.g., 13^2 represents the point $(1, 0, 2)'$ of PG(2, 3)).

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References

[1] D. Bingham, R.R. Sitter, Minimum-aberration two-level fractional factorial split-plot designs, Technometrics 41 (1999) 62–70.
[2] D. Bingham, R.R. Sitter, Design issues in fractional factorial split-plot experiments, J. Quality Tech. 33 (2001) 2–15.
[3] R.C. Bose, Mathematical theory of symmetrical factorial design, Sankhya 8 (1947) 107–166.
[4] H. Chen, C.S. Cheng, Theory of optimal blocking of 2^{n-m} designs, Ann. Statist. 27 (1999) 1948–1973.
[5] H. Chen, A.S. Hedayat, 2^{n-l} designs with weak minimum aberration, Ann. Statist. 24 (1996) 2536–2548.
[6] J. Chen, D.X. Sun, C.F.J. Wu, A catalogue of two-level and three-level fractional factorial designs with small runs, Internat. Statist. Rev. 61 (1993) 131–145.
[7] J. Chen, C.F.J. Wu, Some results on s^{n-k} fractional factorial designs with minimum aberration or optimal moments, Ann. Statist. 19 (1991) 1028–1041.
[8] C.S. Cheng, R. Mukerjee, Regular fractional factorial designs with minimum aberration and maximum estimation capacity, Ann. Statist. 26 (1998) 2289–2300.
[9] A. Dey, R. Mukerjee, Fractional Factorial Plans, Wiley, New York, 1999.
[10] A. Fries, W. Hunter, Minimum aberration 2^{k-p} designs, Technometrics 22 (1980) 601–608.
[11] A. Hedayat, N.J.A. Sloane, J. Stufken, Orthogonal Arrays: Theory and Applications, Springer, Berlin, 1999.
[12] P. Huang, D. Chen, J.O. Voelkel, Minimum-aberration two-level split-plot designs, Technometrics 40 (1998) 314–326.
[13] O. Kempthorne, Design and Analysis of Experiments, Wiley, New York, 1952.
[14] F.J. MacWilliams, N.J.A. Sloane, The Theory of Error-Correcting Codes, North-Holland/Elsevier, Amsterdam, 1998.
[15] R. Mukerjee, K.T. Fang, Fractional factorial split-plot designs with minimum aberration and maximum estimation capacity, Statist. Sinica 12 (2002) 885–904.
[16] C.Y. Suen, H. Chen, C.F.J. Wu, Some identities on q^{n-m} designs with application to minimum aberration designs, Ann. Statist. 25 (1997) 1176–1188.
[17] G. Taguchi, System of Experimental Design, vol. 2, Unipub/Quality Resources, 1987.
[18] B. Tang, C.F.J. Wu, Characterization of minimum aberration 2^{n-k} designs in terms of their complementary designs, Ann. Statist. 24 (1996) 2549–2559.
[19] C.F.J. Wu, M. Hamada, Experiments: Planning, Analysis, and Parameter Design Optimization, Wiley, New York, 2000.